Subduction coefficients of Birman-Wenzl algebras and Racah coefficients of the quantum groups $O_q(n)$ and $Sp_q(2m)$: I. Subduction coefficients

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Subduction coefficients of Birman–Wenzl algebras and Racah coefficients of the quantum groups $O_q(n)$ and $Sp_q(2m)$: I. Subduction coefficients

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Abstract

Irreducible representations of Birman–Wenzl algebras $C_f(r, q)$ in the non-standard basis are discussed. A procedure for evaluating subduction coefficients (SDCs), or the transformation coefficients between standard and non-standard basis of Birman–Wenzl algebras, is formulated based on the linear equation method. SDCs of $C_f(r, q)$ with $f \leqslant 4$ are derived. Racah coefficients of the quantum groups $O_q(n)$ and $Sp_q(2m)$ can be obtained from SDCs of Birman–Wenzl algebras $C_f(r, q)$ by using the Schur–Weyl–Brauer duality relation between Birman–Wenzl algebras $C_f(r, q)$ and the quantum groups $O_q(n)$ and $Sp_q(2m)$.

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1. Introduction

The Birman–Wenzl algebras $C_f(r, q)$, where r and q are two complex parameters, and $f \in \boldsymbol{N}$, were first presented in [1], which is related to some new link polynomials in the knot theory. $C_f(r, q)$ is also a special algebraic realization of braid groups. It has been found that the algebra is useful in connection with the universal $\check{R}$ matrices, which are a class of solutions of the Yang–Baxter equations when the spectral parameter disappears [2–4]. Braid group representations also play an important role in the study of subfactors [5] and in quantum field theory [6, 7]. Most importantly, the Birman–Wenzl algebras $C_f(r, q)$ and the quantum groups of B, C, and D types are in Schur–Weyl–Brauer duality. Let $\mathcal{U}_q$ be a quantum group corresponding to a finite-dimensional complex semisimple Lie algebra of type B_n, C_n, or D_n, and let V be the irreducible representation of $\mathcal{U}_q$ corresponding to the fundamental weight. Then the centralizer algebra $\mathcal{C}_f(\mathcal{U}_q) = \mathrm{End}_{\mathcal{U}_q}(V^{\otimes f})$ is isomorphic to a quotient of the Birman–Wenzl algebra $C_f(q^{n-1}, q)$.

Representations of $C_f(r,q)$ was first studied by Murakami [8], and then by Wenzl [5]. Irreducible representations of $C_f(r,q)$ in the standard basis $C_f(r,q) \supset C_{f-1}(r,q) \supset \cdots \supset C_2(r,q)$ was constructed in [9] by using the induced representation method, and then by Leduc and Ram [10] using the ribbon Hopf algebra approach. In this series of papers, we shall first construct irreducible representations of $C_f(r,q)$ in the non-standard basis for r and q not being roots of unity. A procedure for the evaluation of subduction coefficients (SDCs) of $C_f(r,q) \downarrow C_{f_1}(r,q) \times C_{f_2}(r,q)$ will be proposed based on the linear equation method (LEM). Then, we shall use the powerful Schur–Weyl–Brauer duality relation between $C_f(r,q)$ and the quantum groups $O_q(n)$ or $Sp_q(2m)$ to derive Racah coefficients of $O_q(n)$ and $Sp_q(2m)$ in paper (II), which were never studied before. In this paper, it is assumed that r and q are arbitrary complex numbers, and not roots of unity.

2. The Birman–Wenzl algebras $C_f(r,q)$ in the standard basis

Recall that braid relations among generators g_i $(i = 1, 2, \ldots, f-1)$ of braid group B_f can be written as

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\ g_i g_j &= g_j g_i \qquad \text{for} \quad |i-j| \geqslant 2. \end{aligned} \tag{1}$$

When one more relation, namely the cubic equation

$$(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0 \tag{2}$$

is applied to g_i $(i = 1, 2, \ldots, f-1)$, they generate the Birman–Wenzl algebra $C_f(r,q)$. One knows from [5] that $C_f(r,q)$ is semisimple, and has the same decomposition into full matrix rings as the Brauer algebra D_f discussed in [11–13]. When r and q are complex numbers, which are not roots of unity, one has

$$C_f(r,q) \cong \bigoplus_{\lambda \in \Gamma_f} C_{f,\lambda}(r,q) \tag{3}$$

where $C_{f,\lambda}(r,q)$ is a full matrix ring and Γ_f is the union of the set of all Young diagrams with $f, f-2, f-4, \ldots, 1$ or 0 boxes. If V_λ is a simple $C_{f,\lambda}(r,q)$ module, it decomposes as a $C_{f-1}(r,q)$ module in the form

$$V_\lambda = \bigoplus_{\mu \leftrightarrow \lambda} V_\mu \tag{4}$$

where V_μ is a simple $C_{f-1,\mu}(r,q)$ module, and μ runs over all diagrams that can be obtained by adding or removing one box to or from λ. These facts enable us to construct irreducible representations of $C_f(r,q)$ in the standard basis $C_f(r,q) \supset C_{f-1}(r,q) \supset \cdots \supset C_2(r,q)$ by using the induced representation method [9]. In this case, auxiliary elements e_i for $i = 1, 2, \ldots, f-1$, are helpful in the construction of the basis vectors, which are defined by

$$e_i = 1 - \frac{g_i - g_i^{-1}}{q - q^{-1}}. \tag{5}$$

Basic relations between g_i and e_i are

$$e_i g_i = r^{-1} e_i \qquad e_i g_i^{\pm} e_i = r^{\pm} e_i. \tag{6}$$

It should be noted that Hecke algebra $H_f(q)$ is a subalgebra of $C_f(r,q)$. Therefore, an irrep $[\lambda]$ of $H_f(q)$ is also the same irrep of $C_f(r,q)$ when the corresponding Young diagram of the irrep $[\lambda]$ contains exactly f boxes. Irreducible representations in the standard basis $H_f(q) \supset H_{f-1}(q) \supset \cdots \supset H_2(q)$ of Hecke algebras have been constructed in [5]. These irreps are also the same irreps of $C_f(r,q)$ with $e_i = 0$ for $i = 1, 2, \ldots, f-1$. In the following, we list non-trivial matrix representations of $[\lambda]$ for $C_f(r,q)$ with $f-2, f-4, \ldots, 1$ or 0 boxes, for $f \leqslant 4$, where only upper triangular parts of the matrices are shown because the representation is symmetric, which, along with the matrix representations of Hecke algebras given in [5], will be useful for our purpose.

1. $f = 2$, $\{\lambda\} = \{0\}$ with $\dim\{0\} = 1$. The matrix elements of g_1 and e_1 are

$$g_1 = r^{-1} \qquad e_1 = x \tag{7a}$$

where

$$x = 1 + \frac{r - r^{-1}}{q - q^{-1}}. \tag{7b}$$

2. $f = 3$, $\{\lambda\} = \{1\}$ with $\dim\{1\} = 3$. The matrix elements of g_2 and e_2 are

$$g_2 = \begin{pmatrix} -\frac{q-q^{-1}}{(q^{-1}+r^{-1})(r^{-1}-q)} & -\sqrt{\frac{(r^2-1)(rq^3-1)q^{-2}r^{-1}x^{-1}}{(q^2-1)(qr-1)[2]}} & \sqrt{\frac{q^2(r^2-1)(q^3+r)}{(q^2-1)(q+r)r[2]x}} \\ & \frac{r^{-1}q^{-1}(r^{-1}-q^{-1})}{(r^{-1}-q)[2]} & \frac{r^{-1}}{q[2]}\sqrt{\frac{(q^3r-1)(r+q^3)}{(q+r)(qr-1)}} \\ & & \frac{qr^{-1}(r^{-1}+q)}{[2](q^{-1}+r^{-1})} \end{pmatrix} \tag{8a}$$

$$e_2 = \begin{pmatrix} -\frac{(q-q^{-1})r^{-1}}{(q-r^{-1})(r^{-1}+q^{-1})} & -\sqrt{\frac{(r^2-1)(rq^3-1)r^{-1}x^{-1}}{(q^2-1)(qr-1)[2]}} & -\sqrt{\frac{(r^2-1)(q^3+r)}{(q^2-1)(q+r)rx[2]}} \\ & \frac{(r^2-1)(rq^3-1)}{(q^2-1)(qr-1)r[2]} & \frac{r^2-1}{r[2](q^2-1)}\sqrt{\frac{(q^3r-1)(r+q^3)}{(q+r)(qr-1)}} \\ & & \frac{(r^2-1)(q^3+r)}{[2](q^2-1)r(q+r)} \end{pmatrix} \tag{8b}$$

where the matrix is arranged in the following order of the basis vectors:

$$|1\rangle = \left|\begin{matrix}[1]\\ [0]\end{matrix}\right\rangle \qquad |2\rangle = \left|\begin{matrix}[1]\\ [2]\end{matrix}\right\rangle \qquad |3\rangle = \left|\begin{matrix}[1]\\ [1^2]\end{matrix}\right\rangle. \tag{9}$$

3. $f = 4$, $\{\lambda\} = \{0\}$ with $\dim\{0\} = 3$. The matrix elements of g_3 and e_3 are

$$g_3 = \begin{pmatrix} r^{-1} & & \\ & q & \\ & & -q^{-1} \end{pmatrix} \qquad e_3 = \begin{pmatrix} x & & \\ & 0 & \\ & & 0 \end{pmatrix} \tag{10a}$$

where the matrix is arranged in the following order of the basis vectors:

$$|1\rangle = \left|\begin{matrix}[0]\\ [1]\\ [0]\end{matrix}\right\rangle \qquad |2\rangle \left|\begin{matrix}[0]\\ [1]\\ [2]\end{matrix}\right\rangle \qquad |3\rangle = \left|\begin{matrix}[0]\\ [1]\\ [1^2]\end{matrix}\right\rangle. \tag{10b}$$

4. $f = 4$, $\{\lambda\} = \{2\}$ with $\dim\{2\} = 6$. The matrix elements of g_3 and e_3 are

$$g_3 = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ & \frac{(r-q)r^2q^2(q^2-1)}{(r^2-1)(q^3r-1)} & 0 & \sqrt{\frac{(q^2r^2-1)(1+q^2)(qr-1)(q^5r-1)}{[3]q^4(rq^3-1)^2(r^2-1)}} & \sqrt{\frac{q^2(r-q)(q^3+r)(qr-1)(r^2q^2-1)}{[3](q^3r-1)(r^2-1)^2}} & 0 \\ & & \frac{r^2q^{-1}(q^2-1)}{r^2-1} & 0 & 0 & -\frac{\sqrt{(r^2-q^2)(q^2r^2-1)}}{q(r^2-1)} \\ & & & \frac{q^{-3}r^{-1}(r-q)}{[3](q^3r-1)} & -\sqrt{\frac{(1+q^2)(r-q)(q^3+r)(q^5r-1)}{q^6[3]^2r^2(q^3r-1)(r^2-1)}} & 0 \\ & & & & \frac{(q^2r^2-1)(q^2+1)-qr(q^2-1)}{[3]r(r^2-1)} & 0 \\ & & & & & \frac{1-q^2}{q(r^2-1)} \end{pmatrix} \tag{11a}$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{(q^4-1)r(qr-1)}{q(q^3r-1)(r^2-1)} & 0 & \sqrt{\frac{(1+q^2)(qr-1)(q^5r-1)(q^2r^2-1)}{[3]q^2(rq^3-1)^2(r^2-1)}} & -\sqrt{\frac{(q^2+1)^2(r-q)(q^3+r)(qr-1)(r^2q^2-1)}{[3]q^4(q^3r-1)(r^2-1)^2}} & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{(q^2r^2-1)(q^5r-1)}{(q^2-1)q[3]r(q^3r-1)} & -\sqrt{\frac{(1+q^2)(r-q)(r+q^3)(q^2r^2-1)^2(q^5r-1)}{(q^2-1)^2q^4[3]^2r^2(q^3r-1)(r^2-1)}} & 0 \\ & & & & \frac{(1+q^2)(r-q)(r+q^3)(q^2r^2-1)}{q^3(q^2-1)[3]r(r^2-1)} & 0 \\ & & & & & 0 \end{pmatrix} \tag{11b}$$

where the matrix is arranged in the following order of the basis vectors:

$$\begin{aligned} |1\rangle &= \left|\begin{matrix}[2]\\ [1]\\ [0]\end{matrix}\right\rangle \qquad |2\rangle = \left|\begin{matrix}[2]\\ [1]\\ [2]\end{matrix}\right\rangle \qquad |3\rangle = \left|\begin{matrix}[2]\\ [1]\\ [1^2]\end{matrix}\right\rangle \\ |4\rangle &= \left|\begin{matrix}[2]\\ [3]\end{matrix}\right\rangle \qquad |5\rangle = \left|\begin{matrix}[2]\\ [21]_1\end{matrix}\right\rangle \qquad |6\rangle = \left|\begin{matrix}[2]\\ [21]_2\end{matrix}\right\rangle. \end{aligned} \tag{11c}$$

5. $f = 4,\ \{\lambda\} = \{11\}$ with $\dim\{11\} = 6$. The matrix elements of g_3 and e_3 are

$$g_3 = \begin{pmatrix} -q^{-1} & 0 & 0 & 0 & 0 & 0 \\ & \frac{(q^2-1)r^2}{q(r^2-1)} & 0 & -\frac{\sqrt{(q^2r^2-1)(r^2-q^2)}}{q(r^2-1)} & 0 & 0 \\ & & \frac{(q^2-1)r^2q^{-2}(qr+1)}{(r^2-1)(q^3+r)} & 0 & -\sqrt{\frac{(q+r)(qr+1)(q^3r-1)(r^2-q^2)}{[3]q^6(q^3+r)(r^2-1)^2}} & \sqrt{\frac{(q+r)(r^2-q^2)(q^5+r)(1+q^2)}{[3](q^3+r)^2(r^2-1)}} \\ & & & -\frac{q^2-1}{q(r^2-1)} & 0 & 0 \\ & & & & \frac{(r^2-q^2)(1+q^2)+rq(1-q^2)}{[3]q^4r(r^2-1)} & \sqrt{\frac{(1+q^2)(1+qr)(q^3r-1)(q^5+r)}{q^2r^2(q^3+r)(r^2-1)[3]^2}} \\ & & & & & \frac{q^5(1+qr)}{[3]r(q^3+r)} \end{pmatrix} \tag{12a}$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & \frac{(q^4-1)r(q+r)}{q(q^3+r)(r^2-1)} & 0 & -\frac{1+q^2}{q(r^2-1)}\sqrt{\frac{(q+r)(1+qr)(q^3r-1)(r^2-q^2)}{[3]q^2(q^3+r)}} & -\sqrt{\frac{(1+q^2)(q+r)(q^5+r)(r^2-q^2)}{q^2[3](q^3+r)^2(r^2-1)}} \\ & & & 0 & 0 & 0 \\ & & & & \frac{(1+q^2)(r^2-q^2)(1+qr)(q^3r-1)}{q(q^6-1)r(r^2-1)} & \frac{r^2-q^2}{q^6-1}\sqrt{\frac{(1+q^2)(q^5+r)(1+qr)(q^3r-1)}{r^2(q^3+r)(r^2-1)}} \\ & & & & & \frac{q(q^5+r)(r^2-q^2)}{(q^2-1)q^2[3]r(q^3+r)} \end{pmatrix} \tag{12b}$$

where the matrix is arranged in the following order of the basis vectors:

$$\begin{aligned} |1\rangle &= \left|\begin{matrix}[1^2]\\ [1]\\ [0]\end{matrix}\right\rangle \qquad |2\rangle = \left|\begin{matrix}[1^2]\\ [1]\\ [2]\end{matrix}\right\rangle \qquad |3\rangle = \left|\begin{matrix}[1^2]\\ [1]\\ [1^2]\end{matrix}\right\rangle \\ |4\rangle &= \left|\begin{matrix}[1^2]\\ [3]\end{matrix}\right\rangle \qquad |5\rangle = \left|\begin{matrix}[1^2]\\ [21]_1\end{matrix}\right\rangle \qquad |6\rangle = \left|\begin{matrix}[1^2]\\ [21]_2\end{matrix}\right\rangle. \end{aligned} \tag{12c}$$

3. The Birman–Wenzl algebra $C_f(r, q)$ in the non-standard basis and SDCs of $C_f(r, q) \supset C_{f_1}(r, q) \times C_{f_2}(r, q)$

An irrep of $C_f(r, q)$ is reducible with respect to its subalgebra $C_{f_1}(r, q) \times C_{f_2}(r, q)$ with $f_1 + f_2 = f$. The reduction is denoted by

$$[\lambda]_{f-2k} \downarrow C_{f_1}(r, q) \times C_{f_2}(r, q) = \sum_{\lambda_1\lambda_2} \{\lambda_1\lambda_2\lambda\}([\lambda_1], [\lambda_2]) \tag{13}$$

where it is clearly indicated in the subscript of $[\lambda]$ that there are $f - 2k$ boxes contained in the Young diagram of $[\lambda]$. The orthogonal subduced basis $C_f(r, q) \supset C_{f_1}(r, q) \times C_{f_2}(r, q)$

is called the non-standard basis of $C_f(r,q)$, which follows the same definition [14] of that for Brauer algebras $D_f(n)$. The basis vectors of (13) are denoted by

$$\left|[\lambda]_{f-2k}\ {}^{\tau[\lambda_1]}_{\quad\rho_1}\ {}^{[\lambda_2]}_{\ \rho_2}\right\rangle_{r,q} \tag{14}$$

where $[\lambda_i]\rho_i$ for $i = 1, 2$, can be understood as labels of the standard basis [9] of $C_{f_1}(r,q)$, and $C_{f_2}(r,q)$, respectively, and $\tau = 1, 2, \ldots, \{\lambda_1\lambda_2\lambda\}$ is the multiplicity label needed in the reduction (13).

In order to determine matrix entries of $C_f(r,q)$ in the non-standard basis, one can expand the non-standard basis in terms of the standard basis:

$$\left|[\lambda]_{f-2k}\ {}^{\tau[\lambda_1]}_{\quad\rho_1}\ {}^{[\lambda_2]}_{\ \rho_2}\right\rangle_{r,q} = \sum_{\rho}\left|\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\right\rangle_{r,q}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)}. \tag{15}$$

The expansion coefficient is called the $[\lambda]_{f-2k} \downarrow [\lambda_1] \times [\lambda_2]$ SDC, or the transformation coefficient between the standard and non-standard bases of $C_f(r,q)$. The SDCs satisfy the orthogonality relations:

$$\sum_{\lambda_2\rho_2\tau}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho'\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)} = \delta_{\rho\rho'} \tag{16a}$$

$$\sum_{\rho}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\ \middle|\ \begin{matrix}\tau'[\lambda_1] & [\lambda_2']\\ \rho_1 & \rho_2'\end{matrix}\right\rangle_{(r,q)} = \delta_{\lambda_2\lambda_2'}\delta_{\rho_2\rho_2'}\delta_{\tau\tau'}. \tag{16b}$$

Once the SDCs are known, the matrix elements of $C_f(r,q)$ in the non-standard basis can be derived by using the results of those in the standard basis given in [9, 10].

Similar to the procedure shown in [14], the LEM can also be applied to evaluate SDCs of $C_f(r,q) \supset C_{f_1}(r,q) \times C_{f_2}(r,q)$.

Assume $\{g_1, g_2, \ldots, g_{f_1-1}, e_1, e_2, \ldots, e_{f_1-1}\}$, and $\{g_{f_1+1}, g_{f_1+2}, \ldots, g_{f-1}, e_{f_1+1}, e_{f_1+2}, \ldots, e_{f-1}\}$ are two sets of basic elements of $C_{f_1}(r,q)$, and $C_{f_2}(r,q)$, respectively. By applying $Q_i = g_i$, or e_i with $i = 1, 2, \ldots, f_1 - 1$, and $Q_j = g_j$ or e_j with $j = f_1 + 1, f_1 + 2, \ldots, f - 1$ to (15), and then multiplying the resultants from the left with

$${}_{r,q}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\right|$$

we obtain two sets of linear equations

$$\sum_{\rho_1'}(Q_i)_{\rho_1'\rho_1}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1' & \rho_2\end{matrix}\right\rangle_{(r,q)} = \sum_{\rho'}(Q_i)_{\rho'\rho}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho'\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)} \tag{17a}$$

$$\sum_{\rho_2'}\left(Q_{j-f_1}\right)_{\rho_2'\rho_2}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1 & \rho_2'\end{matrix}\right\rangle_{(r,q)} = \sum_{\rho'}\left(Q_j\right)_{\rho'\rho}\left\langle\begin{matrix}[\lambda]_{f-2k}\\ \rho'\end{matrix}\ \middle|\ \begin{matrix}\tau[\lambda_1] & [\lambda_2]\\ \rho_1 & \rho_2\end{matrix}\right\rangle_{(r,q)} \tag{17b}$$

where $(Q_k)_{\rho'\rho}$ are matrix elements of Q_k in the corresponding standard basis. The linear relations given in (17) or a part of the so-called intertwining relations among SDCs together with the unitarity conditions given in (16) are sufficient in solving these SDCs as has been shown in [14] for the corresponding Brauer algebra case. Using (16) and (17) and the matrix entries of Q_k in the standard basis, one can derive all SDCs for given irreps $[\lambda]_{f-2k}$, $[\lambda_1]$, and $[\lambda_2]$, when the reduction $[\lambda] \downarrow [\lambda_1] \times [\lambda_2]$ is multiplicity-free. It should be noted that the

Table 1. SDCs of $C_3(r,q) \supset C_1(r,q) \times C_2(r,q)$.

$C_3\backslash C_1 \times C_2$	[1] [0]	[1] [2]	[1] [1^2]
[1] [0]	1		
[1] [2]		1	
[1] [1^2]			1

Table 2. SDCs of $C_3(r,q) \supset C_1(r,q) \times C_2(r,q)$.

$C_3\backslash C_1 \times C_2$	[1] [0]	[1] [2]	[1] [1^2]
[1] [0]	$-\frac{(q^2-1)r}{(q+r)(qr-1)}$	$-\sqrt{\frac{(r^2-1)(q^3r-1)}{[2](r+q)(qr-1)^2}}$	$\sqrt{\frac{(r^2-1)(q^3+r)}{[2](qr-1)(q+r)^2}}$
[1] [2]	$\sqrt{\frac{(r^2-1)(q^3r-1)}{[2](r+q)(qr-1)^2}}$	$\frac{r-q}{[2](qr-1)}$	$\sqrt{\frac{(q^3+r)(q^3r-1)}{q^2[2]^2(q+r)(qr-1)}}$
[1] [1^2]	$\sqrt{\frac{(r^2-1)(q^3+r)}{[2](qr-1)(q+r)^2}}$	$-\sqrt{\frac{(q^3+r)(q^3r-1)}{q^2[2]^2(r+q)(qr-1)}}$	$-\frac{1+qr}{[2](r+q)}$

Table 3. SDCs of $C_4(r,q) \supset C_2(r,q) \times C_2(r,q)$.

$C_4\backslash C_2 \times C_2$	[2] [2]	[1^2] [1^2]	[0] [0]
[0] [1] [2]	1		
[0] [1] [1^2]		1	
[0] [1] [0]			1

Table 4. SDCs of $C_4(r,q) \supset C_2(r,q) \times C_2(r,q)$.

$C_4\backslash C_2 \times C_2$	[1^2] [2]	[1^2] [1^2]	[0] [2]
[2] $[21]_2$	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$	$\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$	
[2] [1] [1^2]	$-\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$	
[2] [1] [0]			1

Table 5. SDCs of $C_4(r,q) \supset C_2(r,q) \times C_2(r,q)$.

$C_4\backslash C_2 \times C_2$	[2] [2]	[2] [1^2]	[2] [0]
[2] [3]	$\sqrt{\frac{q(r-q)}{[3](q^3r-1)}}$	$\sqrt{\frac{(q^3+r)(q^5r-1)}{q^2[3](q+r)(q^3r-1)}}$	$\sqrt{\frac{(1+qr)(q^5r-1)}{q[3](q+r)(q^3r-1)}}$
[2] $[21]_1$	$\sqrt{\frac{(q^3+r)(q^5r-1)}{q^4[2][3](r^2-1)}}$	$\sqrt{\frac{(r-q)(q^3r+1)^2}{q^3[2][3](r^2-1)(r+q)}}$	$-\sqrt{\frac{(1+q^2)(r-q)(q^3+r)(1+qr)}{q^3[3](r^2-1)(r+q)}}$
[2] [1] [2]	$\sqrt{\frac{(r-q)(qr+1)(q^5r-1)}{q(1+q^2)(r^2-1)(q^3r-1)}}$	$-\sqrt{\frac{(q^3+r)(qr-1)^2(1+qr)}{q[2](r^2-1)(r+q)(q^3r-1)}}$	$\sqrt{\frac{(q^2-1)^2(1+q^2)r^2}{q(r^2-1)(q+r)(q^3r-1)}}$

SDCs with $k=0$ are the same as those of the corresponding Hecke algebras, of which some examples were provided in [15]. In the multiplicity cases, the relations (17) provides with linearly independent relations for fixed multiplicity label. The corresponding SDCs with the

Table 6. SDCs of $C_4(r,q) \supset C_2(r,q) \times C_2(r,q)$.

$C_4 \backslash C_2 \times C_2$	[2] [1^2]	[2] [2]	[0] [1^2]
[1^2] [1] [2]	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$	$-\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$	
[1^2] [21]$_1$	$\sqrt{\frac{r^2-q^2}{q[2](r^2-1)}}$	$\sqrt{\frac{q^2r^2-1}{q[2](r^2-1)}}$	
[1^2] [1] [0]			1

Table 7. SDCs of $C_4(r,q) \supset C_2(r,q) \times C_2(r,q)$.

$C_4 \backslash C_2 \times C_2$	[1^2] [1^2]	[1^2] [2]	[1^2] [0]
[1^2] [1^3]	$-\sqrt{\frac{q(qr+1)}{q[3](r+q^3)}}$	$-\sqrt{\frac{(q^5+r)(q^3r-1)}{q^2[3](q^3+r)(qr-1)}}$	$\sqrt{\frac{(q^5+r)(r^2-q^2)}{q[3](q+r)(q^3+r)(qr-1)}}$
[1^2] [21]$_2$	$\sqrt{\frac{(q^3r-1)(q^5+r)}{q^4[2][3](r^2-1)}}$	$\sqrt{\frac{(r-q^3)^2(1+qr)}{q^3[2][3](r^2-1)(qr-1)}}$	$\sqrt{\frac{[2](1+qr)(q^3r-1)(r-q)}{q^2[3](r^2-1)(qr-1)}}$
[1^2] [1] [1^2]	$\sqrt{\frac{(r-q)(qr+1)(q^5+r)}{q^2[2](r^2-1)(q^3+r)}}$	$-\sqrt{\frac{(r^2-q^2)(r+q)(q^3r-1)}{(1+q^2)(r^2-1)(qr-1)(q^3+r)}}$	$-\sqrt{\frac{(q^2-1)^2(1+q^2)r^2}{q(r^2-1)(q^3+r)(qr-1)}}$

Table 8. SDCs of $C_4(r,q) \supset C_1(r,q) \times C_3(r,q)$.

$C_4 \backslash C_1 \times C_3$	[1] [1]$_0$	[1] [1]$_2$	[1] [1]$_{1^2}$
[2] [1]$_0$	$\frac{(q^2-1)r}{(q+r)(qr-1)}$	$-\sqrt{\frac{(q^2-1)^2r^2(r-q)^2}{[2](q+r)(qr-1)^2(q^3r-1)(r^2-1)}}$	$\sqrt{\frac{(q^2-1)^2r^2(q^3+r)}{q^2[2](q+r)^2(qr-1)(r^2-1)}}$
[2] [1]$_2$	$-\sqrt{\frac{(r^2-1)(q^3r-1)}{[2](q+r)(qr-1)^2}}$	$\frac{(q^2-1)r(r-q)^2}{[2](qr-1)(q^3r-1)(r^2-1)}$	$\sqrt{\frac{(q^2-1)^2r^2(q^3+r)(q^3r-1)}{q^4[2]^2(q+r)(qr-1)(r^2-1)^2}}$
[2] [1]$_{1^2}$	$-\sqrt{\frac{(r^2-1)(q^3+r)}{[2](q+r)^2(qr-1)}}$	$-\sqrt{\frac{(r-q)^2(q^2-1)^2r^2(q^3+r)}{q^2[2]^2(r^2-1)^2(q+r)(qr-1)(q^3r-1)}}$	$-\frac{(q^2-1)r(qr+1)}{q[2](q+r)(r^2-1)}$
[2] [3]	0	$\sqrt{\frac{[2](qr-1)(q^5r-1)(q^2r^2-1)}{q[3](q^3r-1)^2(r^2-1)}}$	0
[2] [21]$_1$	0	$-\sqrt{\frac{(r-q)(q^3+r)(q^2r^2-1)(qr-1)}{q^2[2]^2[3](r^2-1)^2(q^3r-1)}}$	$-\sqrt{\frac{[3](r^2-q^2)(q^2r^2-1)}{q^2[2]^2(r^2-1)^2}}$
[2] [21]$_2$	0	$-\sqrt{\frac{(r-q)(q^3+r)(q^2r^2-1)(qr-1)}{q^2[2]^2(r^2-1)^2(q^3r-1)}}$	$\sqrt{\frac{(r^2-q^2)(q^2r^2-1)}{q^2[2]^2(r^2-1)^2}}$

fixed multiplicity label can be derived similarly. While the same relations hold for any other multiplicity labels. In order to resolve the multiplicity ambiguity, the SDCs with different multiplicity labels can be chosen to be orthogonal to each other. For example, the SDCs with multiplicity two of $H_6(q) \downarrow H_3(q) \times H_3(q)$ for the reduction $[321] \downarrow [21] \times [21]$ given in [15] are also the SDCs of $C_6(r,q) \downarrow C_3(r,q) \times C_3(r,q)$ for the same irrep. In this case the solution to the SDCs is not unique and depends on the phase convention and the orthogonal basis chosen. One can always make orthogonal transformation to transform one set of SDCs to another within the multiplicity space spanned by the multiplicity labels. For example, the orthogonal transformations for the Wigner coefficients of $SU(n) \supset SU(n-1)$ within the outer-multiplicity space were discussed in detail in [16]. A similar problem in symmetric group was also discussed by McAven *et al* [17]. In this paper, we will only list SDCs of

Table 9. SDCs of $C_4(r,q) \supset C_1(r,q) \times C_3(r,q)$.

$C_4 \backslash C_1 \times C_3$	$[1]\,[3]$	$[1]\,[21]_1$	$[1]\,[21]_2$
$[2]\,[1]_0$	$-\sqrt{\frac{(qr+1)(q^5r-1)}{q[3](q+r)(q^3r-1)}}$	$\sqrt{\frac{(r-q)(q^3+r)(qr+1)}{q^2[3][2](r^2-1)(q+r)}}$	$\sqrt{\frac{(r-q)(q^3+r)(qr+1)}{q^2[2](q+r)(r^2-1)}}$
$[2]\,[1]_2$	$\sqrt{\frac{(q^5r-1)(r-q)^2(qr+1)}{q[2][3](r^2-1)(q^3r-1)^2}}$	$-\sqrt{\frac{(r-q)^3(q^3+r)(qr+1)}{q^2[3][2]^2(q^3r-1)(r^2-1)^2}}$	$\sqrt{\frac{(r-q)(q^3+r)(1+qr)(q^3r-1)}{q^4[2]^2(r^2-1)^2}}$
$[2]\,[1]_{1^2}$	$-\sqrt{\frac{(q^3+r)(q^5r-1)(q^2r^2-1)}{q^3[2][3](r^2-1)(q^3r-1)(r+q)}}$	$\sqrt{\frac{(r-q)(q^3+r)^2(q^2r^2-1)}{q^4[2]^2[3](r^2-1)^2(q+r)}}$	$-\sqrt{\frac{(qr-1)(r-q)(qr+1)^3}{q^2[2]^2(q+r)(r^2-1)^2}}$
$[2]\,[3]$	$\frac{q(r-q)}{[3](q^3r-1)}$	$\sqrt{\frac{[2](r-q)(q^3+r)(q^5r-1)}{q^3[3]^2(q^3r-1)(r^2-1)}}$	0
$[2]\,[21]_1$	$\sqrt{\frac{(r-q)(q^3+r)(q^5r-1)}{q^3[2][3]^2(r^2-1)(q^3r-1)}}$	$\frac{(q^2r^2-1)[2]-r(q^2-1)}{q[2][3](r^2-1)}$	$\sqrt{\frac{[3](q^2-1)^2r^2}{q^2[2]^2(r^2-1)^2}}$
$[2]\,[21]_2$	$\sqrt{\frac{(r-q)(q^3+r)(q^5r-1)}{q^3[2][3](r^2-1)(q^3r-1)}}$	$\frac{(q^2r^2-1)[2]-r(q^2-1)}{q[2](r^2-1)\sqrt{[3]}}$	$-\frac{r(q^2-1)}{q[2](r^2-1)}$

Table 10. SDCs of $C_4(r,q) \supset C_1(r,q) \times C_3(r,q)$.

$C_4 \backslash C_1 \times C_3$	$[1]\,[1]_0$	$[1]\,[1]_2$	$[1]\,[1]_{1^2}$
$[1^2]\,[1]_0$	$-\frac{(q^2-1)r}{(q+r)(qr-1)}$	$-\sqrt{\frac{(q^3r-1)r^2(q^2-1)^2}{q^2[2](q+r)(qr-1)^2(r^2-1)}}$	$\sqrt{\frac{(1+qr)^2r^2(q^2-1)^2}{[2](q+r)^2(qr-1)(q^3+r)(r^2-1)}}$
$[1^2]\,[1]_2$	$\sqrt{\frac{(r^2-1)(q^3r-1)}{[2](q+r)(qr-1)^2}}$	$\frac{(q^2-1)r(r-q)}{[2](qr-1)(1+q^2)(r^2-1)}$	$\sqrt{\frac{(q^2-1)^2r^2(1+qr)^2(q^3r-1)}{q^2[2]^2(q+r)(q^3+r)(qr-1)(r^2-1)^2}}$
$[1^2]\,[1]_{1^2}$	$\sqrt{\frac{(r^2-1)(q^3+r)}{[2](q+r)^2(qr-1)}}$	$-\sqrt{\frac{(1-q^2)^2r^2(q^3r-1)(q^3+r)}{q^4[2]^2(r^2-1)^2(q+r)(qr-1)}}$	$-\frac{(q^2-1)r(qr+1)^2}{[2](q+r)(r^2-1)(q^3+r)}$
$[1^2]\,[21]_1$	0	$\sqrt{\frac{(r^2-q^2)(q^2r^2-1)}{q^2[2]^2(r^2-1)^2}}$	$-\sqrt{\frac{(r^2-q^2)(r+q)(1+qr)(q^3r-1)}{q^2[2]^2(q^3+r)(r^2-1)^2}}$
$[1^2]\,[21]_2$	0	$\sqrt{\frac{[3](r^2-q^2)(q^2r^2-1)}{q^2[2]^2(r^2-1)^2}}$	$-\sqrt{\frac{(r^2-q^2)(r+q)(1+qr)(q^3r-1)}{q^2[2]^2[3](q^3+r)(r^2-1)^2}}$
$[1^2]\,[1^3]$	0	0	$\sqrt{\frac{[2](r^2-q^2)(r+q)(q^5+r)}{q[3](q^3+r)^2(r^2-1)}}$

Table 11. SDCs of $C_4(r,q) \supset C_1(r,q) \times C_3(r,q)$.

$C_4 \backslash C_1 \times C_3$	$[1]\,[21]_1$	$[1]\,[21]_2$	$[1]\,[1^3]$
$[1^2]\,[1]_0$	$\sqrt{\frac{(r-q)(1+qr)(q^3r-1)}{q^2[2](qr-1)(r^2-1)}}$	$-\sqrt{\frac{(r-q)(1+qr)(q^3r-1)}{q^2[2][3](qr-1)(r^2-1)}}$	$\sqrt{\frac{q(r-q)(q^5+r)}{q^2[3](q^3+r)(qr-1)}}$
$[1^2]\,[1]_2$	$-\sqrt{\frac{(q-r)^2(1+qr)(r^2-q^2)}{q^2[2]^2(qr-1)(r^2-1)^2}}$	$-\sqrt{\frac{(r^2-q^2(1+qr)(1-q^3r)^2}{q^4[2]^2[3](qr-1)(r^2-1)^2}}$	$\sqrt{\frac{(r^2-q^2)(q^5+r)(q^3r-1)}{q^3[2][3](qr-1)(r^2-1)(q^3+r)}}$
$[1^2]\,[1]_{1^2}$	$\sqrt{\frac{(r-q)(q^3+r)(1+qr)(q^3r-1)}{q^4[2]^2(r^2-1)^2}}$	$\sqrt{\frac{(r-q)(1+qr)^3(q^3r-1)}{q^2[2]^2[3](r^2-1)^2(q^3+r)}}$	$-\sqrt{\frac{(r-q)(q^5+r)(qr+1)^2}{q[2][3](r^2-1)(q^3+r)^2}}$
$[1^2]\,[21]_1$	$\frac{r(q^2-1)}{(1+q^2)(r^2-1)}$	$-\frac{(r^2-q^2)(1+q^2)-qr(q^2-1)}{q^2[2]\sqrt{[3]}(r^2-1)}$	$\sqrt{\frac{(q^5+r)(1+qr)(q^3r-1)}{q^3[2][3](q^3+r)(r^2-1)}}$
$[1^2]\,[21]_2$	$\frac{\sqrt{3}r(q^2-1)}{(1+q^2)(r^2-1)}$	$\frac{(r^2-q^2)(1+q^2)-qr(q^2-1)}{q^2[2][3](r^2-1)}$	$-\sqrt{\frac{(q^5+r)(1+qr)(q^3r-1)}{q^3[2][3]^2(q^3+r)(r^2-1)}}$
$[1^2]\,[1^3]$	0	$\sqrt{\frac{(1+q^2)(q^5+r)(1+qr)(q^3r-1)}{q^4[3]^2(q^3+r)(r^2-1)}}$	$\frac{q(1+qr)}{[3](q^3+r)}$

$C_f(r,q)$ for the irreps with $k \neq 0$ and $f \leqslant 4$ because $k=0$ SDCs are the same as those of Hecke algebra $H_f(q)$, which are r-independent and have been tabulated in [15].

Similar to the Brauer algebra case shown in [14], the SDCs of $C_f(r,q) \supset C_{f-1}(r,q) \times C_1(r,q)$ are trivial with

$$\left\langle \begin{matrix} [\lambda] \\ \rho \end{matrix} \middle| \begin{matrix} \tau[\lambda_1] & [1] \\ \rho_1 & \end{matrix} \right\rangle_{(r,q)} = \delta_{\rho,[\lambda_1]\rho_1}. \tag{18}$$

Other non-trivial SDCs of $C_f(r,q) \supset C_{f_1}(r,q) \times C_{f_2}(r,q)$ with $k \neq 0$ and $f \leqslant 4$ were derived by using equations (16) and (17), together with the matrix entries of Q_k in the standard basis of $C_f(r,q)$ given in section 2. The phase convention used for the SDCs of $C_f(r,q)$ is the same as that of the Brauer algebra $D_f(n)$ shown in [14]. The package Mathematica was implemented in the formalized algorithm based on our procedure. The results are listed in tables 1–11.

4. Conclusions

In this paper, the Birman–Wenzl algebras $C_f(r,q)$ in the non-standard basis are discussed. The SDCs of $C_f(r,q) \supset C_{f_1}(r,q) \times C_{f_2}(r,q)$ with $f_1 + f_2 = f$ and $f \leqslant 4$ are derived by using the LEM. The SDCs of $C_f(r,q)$ will be useful in evaluating Racah coefficients of $O_q(n)$ and $Sp_q(2m)$ by using the Schur–Weyl–Brauer duality relation between the Birman–Wenzl algebras with $r = q^{n-1}$ and the corresponding quantum groups of B, C, D types when q is not a root of unity, which have never been studied. The Racah coefficients will be discussed in our next paper.

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References

[1] Birman J and Wenzl H 1989 *Trans. Am. Math. Soc.* **313** 249
[2] Mackay N J 1992 *J. Math. Phys.* **33** 1529
[3] Grimm U 1994 *Lett. Math. Phys.* **32** 183
[4] Ge M L, Liu G and Xue K 1991 *J. Phys. A: Math. Gen.* **24** 2679
[5] Wenzl H 1990 *Commun. Math. Phys.* **133** 383
[6] Fredenhagen K, Rehren K H and Schroer B 1989 *Commun. Math. Phys.* **125** 201
[7] Felder G, Fröhlich J and Keller G 1990 *Commun. Math. Phys.* **128** 647
[8] Murakami J 1990 *Publ. RIMS. Kyoto Univ.* **26** 935
[9] Feng Pan 1995 *Sci. China* A **26** 1344
[10] Leduc R and Ram A 1997 *Adv. Math.* **125** 1
[11] Brauer R 1937 *Ann. Math.* **63** 854
[12] Wenzl H 1989 *Ann. Math.* **128** 173
[13] Feng Pan 1995 *J. Phys. A: Math. Gen.* **28** 3139
[14] Feng Pan and Lianrong Dai 1996 *J. Phys. A: Math. Gen.* **29** 5079
[15] Feng Pan and Chen J Q 1993 *J. Phys. A: Math. Gen.* **26** 4299
[16] Feng Pan and Draayer J P 1998 *J. Math. Phys.* **34** 5642
[17] McAven L F, Butler P H and Hamel A M 1998 *J. Phys. A: Math. Gen.* **31** 8363